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INFLUENCE OF RELATIVE PLY ORIENTATIONS ON THE NATURE OF THE EDGE EFFECT SINGULARITIES FOR A CIRCULAR HOLE IN A COMPOSITE LAMINATE

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Abstract—A method is developed for the analysis of the nature of the singularities at the free edge of an elliptical hole in a composite laminate. The method is general enough to be applicable to any type of laminate with or without a hole. Boundary layer theory, as originated in aerodynamics is used to simplify the equations applied within the boundary layer region, while compatibility is achieved at the other border of that region to comply with anisotropic plate theory. In the present case the method is applied to investigate the effect of the free edge of a hole on stresses at the ply interface within the boundary layer. The immediate application within that region is the solution for the Energy Release Rate which is essential to the evaluation of critical conditions such as delamination initiation. Attention is focused on the influence of adjacent ply orientations on the order of singularity at the interface.

NOTATION

A	coefficients matrix
A_i	constants of integration
a_i	coefficients of polynomials in the particular solution
a	distance from hole to laminate edge
B	cross sectional area at cut perpendicular to load direction
B_F	free edge surface
B_I	interface plane
$\mathbb{B}$	results vector of set of linear equations
b	out-of-plane coordinate of upper surface of laminate
$\mathbb{C}$	coefficients vector, final description
c_i	coefficients vector of homogeneous solution
c	out-of-plane coordinate of lower surface of laminate
D	vector of unknowns in the complete formulation
$\mathbb{D}$	domain of integration
d_1, d_3	elliptical hole axes in x_1 and x_3 directions, respectively
$d_n^{(h)}$	coefficients vector of the homogeneous solution
ds	infinitesimal length
E_1, E_2, E_3	unidirectional material properties
e	distance from hole to laminate edge
F	Lekhnitskii's stress potentials
F_k	stress function used for the homogeneous solution
$f_{\alpha n}$	stress eigenfunctions of the homogeneous solution
G_{ij}	functions of the compliance and coeff. of part. solution
$g_{\beta n}$	displacements eigenfunctions of the homogeneous solution
$(\ )^{(h)}$	magnitude related to the homogeneous solution
K_i	near-field parameters
$(\ )_k$	magnitude related to the characteristic equation
L_i	differential operator
l_i	differential operator
M_{ij}	remote loading-moment w.r.t. the indices
$(\ )^{(m)}$	magnitude related to the mth ply
n	number of unknowns ≡ number of columns in the coefficients matrix
p_k	function related to the homogeneous solution
$(\ )^{(p)}$	magnitude related to the particular solution
q_k	function related to the homogeneous solution
P_{33}	uniaxial remote load in the x_3 direction
p	order of matrix—number of rows
$\mathbb{R}$	residual when utilizing a weighting technique
r	polar coordinate starts at the origin of the Cartesian system
S_{ij}	tensorial compliance matrix
$\tilde{S}_{ij}$	reduced tensorial compliance matrix
S.V.D.	the method of singular value decomposition

$(\ )^{(T)}$	transpose
t	laminate thickness
t_k	function related to the homogeneous solution
U	matrix of orthonormalized eigenvectors
U_i	function related to the process of governing equations
u_i	displacements vector
u_{i0}	rigid body displacements
V	matrix of orthonormalized eigenvectors
v	vector related to the singular value decomposition method
w	vector related to the singular value decomposition method
x_i	directions
Z_k	argument of Lekhnitskii's stress functions
α	scanning azimuth at the hole
$(\ )_\alpha$	Greek notation: 1–6
δ_i	series of eigenvalues
$\delta_i^{(m)}$	series of eigenvalues at interface m
ε_{ij}	strain tensor
η_k	coefficients related to the characteristic equation
Ψ	Lekhnitskii's stress potential
ϕ	trial function related to the weighting technique
μ_k	roots of the characteristic equation
ν_{ij}	Poisson's ratio
$\Theta^{(m)}$	fiber direction at mth ply
$\theta^{(m)}$	polar coordinate starts from the examined interface
Σ	diagonal matrix of eigenvalues, related to the S.V.D.
σ_i	stresses in Greek notation
σ_{ij}	stress tensor
$\sigma_{ij\,\mathrm{cal}}$	stress tensor computed for the anisotropic plate with hole
φ	angle measured from the interface to B_1
ω_i	rigid body rotations
$(\bar{\ })$	conjugate of complex number
$(\ )', (\ )''$	first and second differentiation
∂	partial differentiation
$+$	transpose of the conjugates
$\det|\ |$	determinant

1. INTRODUCTION

It is known that a wide range of properties and performance can be achieved through the utilization of composite laminates in structures, since it is possible to adjust the ply orientations, stacking sequence, and thickness. At the edge of the laminate, however, high values of stresses are often obtained due to ply deformation mismatching, which may lead to delamination. That delamination can be controlled by artificial means such as stitching, tufting, or constraining the edge by caps. A more natural procedure is available through optimization of fiber orientation, weaving, and layer sequencing in the vicinity of the edge in order to reduce edge effects, or even by reversing the interlaminar situation by changing tension conditions to compression.

The behavior of the stress field at the edges of composite laminates due to deformation mismatching has been the subject of extensive investigation during the last two decades. Pipes and Pagano (1970) used a finite difference method to solve the relevant elasticity equations, whereas Wang and Crossman (1977) employed a finite element approach to investigate this phenomenon. Recently, an approximate method was presented by Kassapoglou and Lagace (1987) using the force balance method in conjunction with minimization of energy. Due to the approximate nature of the approaches involved in the previous studies, it was not possible to determine the order of singularity of the stresses at the free edges. Wang and Choi (1982a,b) derived an analytical solution, following a Lekhnitskii (1963) formulation, and obtained the exact order of singularity at the edge of the laminate. Their derivation involves a special form of Lekhnistkii's stress potentials which explicitly includes a parameter identified as the order of the singularity. However, their solution for the stresses is approximate, and due to mathematical difficulties the method was only applied to the analysis of special types of laminates at the straight free edge, as was done by the previously mentioned researchers.

Zwiers *et al.* (1982) have used the method of Stroh (1962) to consider the problem, and find a logarithmic singularity in addition to the r^δ singularity for general laminate

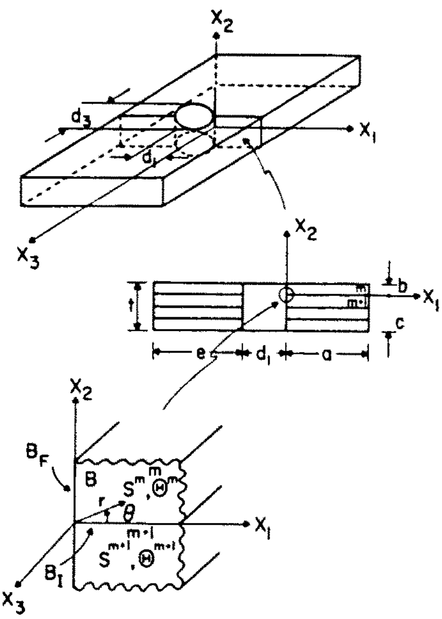

Fig. 1. Geometry and coordinates of composite laminate with elliptical hole.

interfaces in some cases. However, the nature of ln (r) singularity does not depend on the ply orientations on either side of the interface. Hence, it is not of interest for the present analysis which seeks to determine the manner in which the relative ply orientations influence the nature of the singularity as a basis for predicting the interface which is most likely to delaminate.

In the present paper the approach of Wang and Choi (1982a,b) is followed for the analysis of a laminated plate with an elliptical hole. The proposed method is general in the sense that it can be applied to any type of composite laminate with an elliptical hole. For the special case when the effect of the hole is disregarded, the general method is applicable to the special types of laminates treated by Wang and Choi (1982a,b). The analysis leads to an over-determined system of equations which shows ill-conditioned behavior. This mathematical obstacle is overcome by adopting the Singular Value Decomposition Method (Stewart, 1973) to determine the real rank of the matrix when it is less than full rank.

The governing equations consist of differential equations and therefore are decomposed into homogeneous and particular parts. The derivation of the homogeneous solution is identical to the method of Wang and Choi (1982a,b). The particular solution satisfies the governing equations, free edge conditions, the interfacial continuity relations, and the upper and lower traction-free surface requirements, and represents the influence of the hole. The effect of an elliptical hole is incorporated by adopting Lekhnitskii's solution for anisotropic plates containing an elliptical cavity. By conversion of the composite laminate into an anisotropic plate via classical lamination theory (Jones, 1975), the strains were evaluated from Lekhnitskii's solution. The corresponding stresses in the different plies can thus be determined. As the analysis is based on a set of eigenvalues and includes some numerical integration, the accuracy of the results was assessed and demonstrated by selecting various numbers of eigenvalues and points of integration. Results are given which exhibit the effect of the hole's edge on the behavior of stresses within the boundary layer and, for special cases, throughout the laminate.

2. BASIC FORMULATION

2.1. *Solution methodology*

Consider a composite laminate containing an elliptical cutout as shown in Fig. 1. A system of Cartesian coordinates is introduced whose origin is located at the bore edge at the examined interface and its orientation follows the scanning azimuth in such a way that the x_1 axis is always radially oriented away from the center of the hole. Every lamina is assumed to be elastic and anisotropic, obeying the generalized Hooke's law:

$$\{\varepsilon\} = [S]\{\sigma\} \tag{1}$$

in which $\{\varepsilon\}$ are the strain components of the mth ply,

$$\{\varepsilon\} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}]^T.$$

The solution methodology follows Lekhnitskii's (1963) approach for anisotropic plates as used by Wang and Choi (1982a,b). The problem is treated as a boundary layer problem. It is somewhat similar to the aerodynamic problem of viscous flow around an airfoil. In the aerodynamic problem, we consider a streaming flow past a slender body. The fluid viscosity is taken to be relatively small and the shearing stresses developed are very small. It is known that except for a thin layer adjacent to the solid body, the transverse velocity gradients are negligibly small throughout the flow field. However, within that thin boundary layer, large shearing velocities are produced resulting in large shear stresses. The importance of this concept is that it allows us to apply the more complicated equations related to the boundary only within that thin layer, and some appreciable simplifying assumptions can reasonably be made. In the aerodynamics case, these are the viscous motion equations. In our case, the boundary layer is a relatively thin region in the vicinity of the edge. At the free edge the stress field is singular and thus its values are infinite. Within the boundary layer stresses change rapidly from the edge to the other side of the region where they agree with results from classical lamination theory (Jones, 1975) and anisotropic plates (Lekhnitski, 1963). Within the boundary layer, changes with respect to x_1 and x_2 are considered to be larger than changes with respect to x_3. Thus, within that region we may simplify the problem and neglect variations with respect to x_3 while requiring compatability with the above-mentioned solutions which take changes with respect to x_3 into account, such as the solution of an anisotropic plate with a cavity (Lekhnitskii, 1963). Several assumptions should be noted.

(i) The composite laminate is of finite width.

(ii) The laminate is long enough such that end effects can be neglected.

(iii) Due to the neglect of variations with respect to x_3, we may assume a state of generalized plane deformation within the boundary layer.

The equilibrium equations, in the absence of body forces, are given by

$$\sigma_{ij,j} = 0 \quad i, j = 1, 2, 3. \tag{2}$$

Due to assumption 3, derivatives with respect to x_3 are omitted, reducing (2) to

$$\sigma_{11,1} + \sigma_{12,2} = 0 \tag{3a}$$

$$\sigma_{21,1} + \sigma_{22,2} = 0 \tag{3b}$$

$$\sigma_{31,1} + \sigma_{32,2} = 0. \tag{3c}$$

The small strain tensor is given in terms of the displacements u_i by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{4}$$

Using (4) in (1) and integrating provides

$$u_1 = -\tfrac{1}{2}A_1 S_{33} x_3^2 - A_4 x_2 x_3 + U_1(x_1, x_2) + \omega_2 x_3 - \omega_3 x_2 + u_{10} \tag{5a}$$

$$u_2 = -\tfrac{1}{2}A_2 S_{33} x_3^2 + A_4 x_1 x_3 + U_2(x_1, x_2) + \omega_3 x_1 - \omega_1 x_3 + u_{20} \tag{5b}$$

$$u_3 = (A_1 x_1 + A_2 x_2 + A_3) S_{33} x_3 + U_3(x_1, x_2) + \omega_1 x_2 - \omega_2 x_1 + u_{30} \tag{5c}$$

where u_{i0} and ω_i are rigid-body translations and rotations, respectively. The stress in the longitudinal direction, σ_{33} is given by

$$\sigma_{33} = A_1x_1 + A_2x_2 + A_3 - \frac{S_{3j}\sigma_j}{S_{33}} \tag{6}$$

where $j = 1, 2, 4, 5, 6$ using the contracted Greek notation for stresses.

The derivatives of functions $U_1(x_1, x_2)$, $U_2(x_1, x_2)$, $U_3(x_1, x_2)$ are expressed in the form

$$U_{1,1} = \tilde{S}_{1j}\sigma_j + S_{13}(A_1x_1 + A_2x_2 + A_3) \tag{7a}$$

$$U_{2,2} = \tilde{S}_{2j}\sigma_j + S_{23}(A_1x_1 + A_2x_2 + A_3) \tag{7b}$$

$$U_{3,1} = \tilde{S}_{5j}\sigma_j + S_{53}(A_1x_1 + A_2x_2 + A_3) + A_4x_2 \tag{7c}$$

$$U_{3,2} = \tilde{S}_{4j}\sigma_j + S_{43}(A_1x_1 + A_2x_2 + A_3) - A_4x_1 \tag{7d}$$

$$U_{1,2} + U_{2,2} = \tilde{S}_{6j}\sigma_j + S_{63}(A_1x_1 + A_2x_2 + A_3) \tag{7e}$$

where $\tilde{S}$ is the reduced form of S given by

$$\tilde{S}_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}} \quad i, j = 1, 2, 4, 5, 6. \tag{8}$$

Following Wang and Choi (1982a,b), we adopt Lekhnitskii's stress potentials F, Ψ defined by

$$\sigma_1 = \sigma_{11} = F_{,22}; \quad \sigma_2 = \sigma_{22} = F_{,11}; \quad \sigma_4 = \sigma_{23} = -\Psi_{,1};$$
$$\sigma_5 = \sigma_{13} = \Psi_{,2}; \quad \sigma_6 = \sigma_{12} = -F_{,12}. \tag{9}$$

Equation (9) satisfies (3) and when used in conjunction with (5) and (7) it creates a system of governing equations that can be expressed in terms of partial differential operators which have the form

$$\begin{cases} L_3F + L_2\Psi = -2A_4 + A_1S_{34} - A_2S_{35} \\ L_4F + L_3\Psi = 0 \end{cases} \tag{10}$$

where

$$L_2 = \tilde{S}_{44}\frac{\partial^2}{\partial x_1^2} - \tilde{S}_{45}\frac{\partial^2}{\partial x_1\,\partial x_2} - \tilde{S}_{55}\frac{\partial^2}{\partial x_2^2} \tag{11a}$$

$$L_3 = -\tilde{S}_{24}\frac{\partial^3}{\partial x_1^3} + (\tilde{S}_{25} + \tilde{S}_{46})\frac{\partial^3}{\partial x_1^2\,\partial x_2} + (\tilde{S}_{14} + \tilde{S}_{56})\frac{\partial^3}{\partial x_1\,\partial x_2^2} + \tilde{S}_{15}\frac{\partial^3}{\partial x_2^3} \tag{11b}$$

$$L_4 = -\tilde{S}_{22}\frac{\partial^4}{\partial x_1^4} - 2\tilde{S}_{26}\frac{\partial^4}{\partial x_1^3\,\partial x_2} + (2\tilde{S}_{12} + \tilde{S}_{66})\frac{\partial^4}{\partial x_1^2\,\partial x_2^2} - 2\tilde{S}_{16}\frac{\partial^4}{\partial x_1\,\partial x_2^3} + \tilde{S}_{11}\frac{\partial^4}{\partial x_2^4}. \tag{11c}$$

2.2. *Boundary conditions*

We consider three types of boundary conditions, as follows in the next sections.

2.2.1. *Traction-free edge boundary condtions.* Assuming that the edges of the laminate and the hole are traction-free, it follows that

$$\sigma_{11} = \sigma_{13} = \sigma_{12} = 0 \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = 0. \tag{12}$$

2.2.2. *End conditions.* We require static equilibrium with the remote loading by forming the following integrals over the cross-sectional area B as shown in Fig. 1 (Lekhnitskii, 1963)

$$\int_B\int \sigma_{13}\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \tag{13a}$$

$$\int_B\int \sigma_{23}\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \tag{13b}$$

$$\int_B\int \sigma_{33}\,\mathrm{d}x_1\,\mathrm{d}x_2 = P_{33} \tag{13c}$$

$$\int_B\int \sigma_{33}x_2\,\mathrm{d}x_1\,\mathrm{d}x_2 = M_{11} \tag{13d}$$

$$\int_B\int \sigma_{33}x_1\,\mathrm{d}x_1\,\mathrm{d}x_2 = M_{22} \tag{13e}$$

$$\int_B\int (\sigma_{23}x_1 - \sigma_{13}x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = M_{12}. \tag{13f}$$

For cases where the analysis is done at an azimuthal angle α different than 0°, the coordinate system is rotated such that the x_1 axis is tangent to the hole surface and creates an angle α with respect to the longitudinal axis of the plate. The domain in which integration is carried out is the cross-sectional area $B/\cos(\alpha)$.

2.2.3. *The cavity boundary conditions.* A special treatment is required in the vicinity of the hole. This is achieved by conversion of the laminate plate cross section into an anisotropic plate via its effective elastic constants (E_x, E_y, G_{xy}, v_{xy}) calculated by classical lamination theory (Jones, 1975). Analysis of such a plate with cavity subjected to various loads, is done following Lekhnitskii (1963). The resulting strains, when multiplied by the stiffness components of the relevant ply, provide the planar stress distribution in that ply. These stress distributions are applicable away from the hole where the edge effects are negligible.

2.3. *Interfacial continuity*

Continuity of tractions must be satisfied at the interface between the m and $m+1$ plies:

$$\sigma_{2i}^{(m)} = \sigma_{2i}^{(m+1)} \quad i = 1, 2, 3; \quad x_2 = 0; \tag{14}$$

and the displacements must be continuous:

$$u_i^{(m)} = u_i^{(m+1)} \quad i = 1, 2, 3; \quad x_2 = 0 \tag{15}$$

3. SOLUTION OF THE GOVERNING EQUATIONS

The solution consists of two parts, homogeneous and particular solutions. The homogeneous part can be exactly derived and provides the stress singularities at the edges. On the other hand, the particular solution can not be exactly obtained, and an approximate method is applied.

3.1. *The homogeneous solution*

Following Lekhnitskii (1963), the general forms of his stress potential are taken in the form

$$F(x_1, x_2) = \sum_{k=1}^{6} F_k(Z_k) \tag{16a}$$

$$\Psi(x_1, x_2) = \sum_{k=1}^{6} \eta_k F'_k(Z_k) \tag{16b}$$

where $Z_k = x_1 + \mu_k x_2$, μ_k are the roots of the characteristic equation as shown below, η_k are ratios of components of the characteristic equation, and $F'_k(Z_k)$ is the derivative w.r.t. the argument Z_k.

Regarding the homogeneous form of eqn (10), we consider the potentials F and Ψ to consist of two parts denoted by indices 1 and 0 designating the homogeneous and the particular solutions, respectively. The characteristic equation of the homogeneous solution is defined by the left-hand side (10). Eliminating one of the functions, say Ψ_1, we obtain a 6th order equation for the remaining F_1,

$$(L_4 L_2 - L_3^2)F_1 = 0 \tag{17a}$$

which can be decomposed into

$$D_6 D_5 D_4 D_3 D_2 D_1 F_1 = 0 \tag{17b}$$

where

$$D_k = \frac{\partial}{\partial x_2} - \mu_k \frac{\partial}{\partial x_1}.$$

Consequently, new sets of operators are obtained from (11):

$$l_2 = \tilde{S}_{55}\mu^2 - 2\tilde{S}_{45}\mu + \tilde{S}_{44} \tag{18a}$$

$$l_3 = \tilde{S}_{15}\mu^3 - (\tilde{S}_{14} + \tilde{S}_{56})\mu^2 + (\tilde{S}_{25} + \tilde{S}_{46}) - \tilde{S}_{24} \tag{18b}$$

$$l_4 = \tilde{S}_{11}\mu^4 - 2\tilde{S}_{16}\mu^3 + (2\tilde{S}_{12} + \tilde{S}_{66})\mu^2 - 2\tilde{S}_{26}\mu + \tilde{S}_{22}. \tag{18c}$$

The resulting characteristic equation is

$$l_4(\mu)l_2(\mu) - l_3^2(\mu) = 0 \tag{19a}$$

also

$$\eta_k = -\frac{l_3(\mu_k)}{l_2(\mu_k)} = -\frac{l_4(\mu_k)}{l_3(\mu_k)}. \tag{19b}$$

It has been shown by Lekhnitskii (1963) that the μ_k are complex conjugates where the real part vanishes for orthotropic materials. Solving the polynomial (19), and substituting for the stresses and displacements (9) and (7), respectively, yields the following results designated by (h) for the homogeneous part of the solution

$$\sigma_{11}^{(h)} = \sum_{k=1}^{6} \mu_k^2 F''(Z_k) \tag{20a}$$

$$\sigma_{22}^{(h)} = \sum_{k=1}^{6} F''(Z_k) \tag{20b}$$

$$\sigma_{23}^{(h)} = -\sum_{k=1}^{6} \eta_k F''(Z_k) \tag{20c}$$

$$\sigma_{13}^{(h)} = \sum_{k=1}^{6} \eta_k \mu_k F''(Z_k) \tag{20d}$$

$$\sigma_{12}^{(h)} = -\sum_{k=1}^{6} \mu_k F''(Z_k) \tag{20e}$$

$$u_1^{(h)} = \sum_{k=1}^{6} p_k F'(Z_k) \tag{20f}$$

$$u_2^{(h)} = \sum_{k=1}^{6} q_k F'(Z_k) \tag{20g}$$

$$u_3^{(h)} = \sum_{k=1}^{6} t_k F'(Z_k) \tag{20h}$$

where

$$p_k = \tilde{S}_{11}\mu_k^2 + \tilde{S}_{12} - \tilde{S}_{14}\eta_k + \tilde{S}_{15}\eta_k\mu_k - \tilde{S}_{16}\mu_k \tag{21a}$$

$$q_k = \tilde{S}_{12}\mu_k + \frac{\tilde{S}_{22}}{\mu_k} - \frac{\tilde{S}_{24}\eta_k}{\mu_k} + \tilde{S}_{25}\eta_k - \tilde{S}_{26} \tag{21b}$$

$$t_k = \tilde{S}_{14}\mu_k + \frac{\tilde{S}_{24}}{\mu_k} - \frac{\tilde{S}_{44}\eta_k}{\mu_k} + \tilde{S}_{45}\eta_k - \tilde{S}_{46}. \tag{21c}$$

Following the idea of Wang and Choi (1982a,b), the functions $F_k(Z_k)$ are expressed in the form

$$F_k(Z_k) = c_k \frac{Z_k^{\delta+2}}{(\delta+1)(\delta+2)}. \tag{22}$$

By choosing this particular expression, it can be readily shown (by performing second-order derivatives) that the general form of the stresses can be represented in the form

$$\sigma_i = K_i r^{\delta 1}. \tag{23}$$

It is obvious that, by solving for δ, we obtain the exact order of the singularity as r approaches zero. Using a different approach, Zwiers *et al.* (1982) have found that a complete solution to this problem involves an additional term which represents the logarithmic behavior of the singularity as well as dependence of a constant taken to be a material parameter. This part of the behavior and the associated terms are of no importance to the present study since they do not vary with changes in adjacent ply orientations for a given laminate. For the present solution scheme, we require that (22) followed by (23) will satisfy all boundary conditions and governing equations for the homogeneous and particular parts of the solution. Substituting (22) into (20) provides the following:

$$\sigma_{11}^{(h)} = \sum_{k=1}^{3} [c_k \mu_k^2 Z_k^\delta + c_{k+3} \bar{\mu}_k^2 \bar{Z}_k^\delta] \tag{24a}$$

$$\sigma_{22}^{(h)} = \sum_{k=1}^{3} [c_k Z_k^\delta + c_{k+3} \bar{Z}_k^\delta] \tag{24b}$$

$$\sigma_{23}^{(h)} = -\sum_{k=1}^{3} [c_k \eta_k Z_k^\delta + c_{k+3} \bar{\eta}_k \bar{Z}_k^\delta] \tag{24c}$$

$$\sigma_{13}^{(h)} = \sum_{k=1}^{3} [c_k \eta_k \mu_k Z_k^{\delta} + c_{k+3} \bar{\eta}_k \bar{\mu}_k \bar{Z}_k^{\delta}] \tag{24d}$$

$$\sigma_{12}^{(h)} = -\sum_{k=1}^{3} [c_k \mu_k Z_k^{\delta} + c_{k+3} \bar{\mu}_k \bar{Z}_k^{\delta}] \tag{24e}$$

$$u_1^{(h)} = \sum_{k=1}^{3} [c_k p_k Z_k^{\delta+1} + c_{k+3} \bar{p}_k \bar{Z}_k^{\delta+1}]/(\delta+1) \tag{24f}$$

$$u_2^{(h)} = \sum_{k=1}^{3} [c_k q_k Z_k^{\delta+1} + c_{k+3} \bar{q}_k \bar{Z}_k^{\delta+1}]/(\delta+1) \tag{24g}$$

$$u_3^{(h)} = \sum_{k=1}^{3} [c_k t_k Z_k^{\delta+1} + c_{k+3} \bar{t}_k \bar{Z}_k^{\delta+1}]/(\delta+1). \tag{24h}$$

It should be noted that the present contribution (23) from the homogeneous solution involves the parameter δ. This parameter depends on the specific geometry in the close vicinity of the edge as well as on the elastic constants of the two adjacent plies. Thus, (23) is valid at the hole as well as at the plate edge, and refers to the relevant ply pair in which δ was calculated.

Substituting (22) into the free edge boundary conditions (12), yields three equations for each of the two adjacent plies, resulting in a total of six equations. Similarly, substitution into the interfacial conditions (15) contributes an additional six equations. There are six unknown coefficients c_k, $k = 1$-6, for each layer and the additional unknown power δ. This system of 12 algebraic equations can be presented in a matrix form,

$$[A]\{C\} = 0 \tag{25}$$

where $[A]$ is a 12×12 matrix whose elements involve δ as a power. In addition,

$$\{c\} = [c_k^{(m)}, c_k^{(m+1)}]^T \quad k = 1,2,3,4,5,6.$$

This system establishes a nonlinear eigenvalue problem for which $\{C\}$ are the eigenvectors and δ_i are the corresponding eigenvalues determined from the requirement that $[A]$ must vanish for a non-trivial solution:

$$\det[A] = 0. \tag{26}$$

The solution of eqn (26) is performed by a deflation technique as presented by Muller (1956). Since (26) is a transcendental equation, an infinite set of solutions for δ is obtained. The algebraically smallest eigenvalue is a real number in $[-1,0]$ and is the order of the singularity as explained by Wang and Choi (1982a,b). For the case of an angle-ply laminate, the higher eigenvalues are either integers or pairs of conjugate complex numbers. The properly truncated set of eigenvalues is used in the particular solution to ensure convergence. Once (25) is solved, the stresses and displacements are obtained from (24) using the expressions

$$\sigma_{\alpha}^{(h)} = \sum_n d_n^{(h)} f_{\alpha n}(x_1, x_2; \delta_n) \quad \alpha = 1,2,4,5,6 \tag{27a}$$

$$u_{\beta}^{(h)} = \sum_n d_n^{(h)} g_{\beta n}(x_1, x_2; \delta_n) \quad \beta = 1,2,3 \tag{27b}$$

$$\sigma_3^{(h)} = -\frac{S_{3j}\sigma_j^{(h)}}{S_{33}} \tag{27c}$$

where $f_{\alpha n}$ and $g_{\beta n}$ are the eigenfunctions which coincide with the right-hand side of eqn (24) and include the infinite set of δ_n. The infinite set of coefficients $\{d_n^{(h)}\}$ is to be determined in conjunction with the particular solution.

3.2. *The particular solution*

A particular solution (denoted by superscript p) to eqn (10) is expressed in the form

$$F^{(p)} = a_1x_1^3+a_2x_1^2x_2+a_3x_1x_2^2+a_4x_2^3+a_5x_1^2+a_6x_1x_2+a_7x_2^2$$
$$\Psi^{(p)} = a_8x_1^2+a_9x_1x_2+a_{10}x_2^2+a_{11}x_1+a_{12}x_2. \quad (28)$$

Substitution of eqn (28) into eqn (9) yields

$$\sigma_{11}^{(p)} = 2a_3x_1+6a_4x_2+2a_7 \quad (29a)$$
$$\sigma_{22}^{(p)} = 6a_1x_1+2a_2x_2+2a_5 \quad (29b)$$
$$\sigma_{23}^{(p)} = -2a_8x_1-a_9x_2-a_{11} \quad (29c)$$
$$\sigma_{13}^{(p)} = a_9x_1+2a_{10}x_2+a_{12} \quad (29d)$$
$$\sigma_{12}^{(p)} = -2a_2x_1-2a_3x_2-a_6 \quad (29e)$$

and

$$\sigma_{33}^{(p)} = (A_1x_1+A_2x_2+A_3)-\frac{S_{3j}\sigma_j^{(p)}}{S_{33}}. \quad (30)$$

The expression of the particular part of the displacements, $u_i^{(p)}$, follows exactly the form of eqns (5a–c), in which

$$U_1^{(p)} = \tfrac{1}{2}G_{11}x_1^2+G_{12}x_1x_2+G_{13}x_1+\tfrac{1}{2}(G_{62}-G_{21})x_2^2+\tfrac{1}{2}G_{63}x_2 \quad (31a)$$
$$U_2^{(p)} = G_{12}x_1x_2+\tfrac{1}{2}G_{22}x_2^2+G_{23}x_2+\tfrac{1}{2}(G_{61}-G_{12})x_1^2+\tfrac{1}{2}G_{63}x_1 \quad (31b)$$
$$U_3^{(p)} = \tfrac{1}{2}G_{51}x_1^2+(G_{52}+A_4)x_1x_2+G_{53}x_1+\tfrac{1}{2}G_{42}x_2^2+G_{43}x_2 \quad (31c)$$

and

$$G_{j1} = 2\tilde{S}_{j1}a_3+6\tilde{S}_{j2}a_1-2\tilde{S}_{j4}a_8+\tilde{S}_{j5}a_9-\tilde{S}_{j6}a_2+S_{j3}A_1 \quad (32a)$$
$$G_{j2} = 6\tilde{S}_{j1}a_4+2\tilde{S}_{j2}a_2-\tilde{S}_{j4}a_9+2\tilde{S}_{j5}a_{10}-2\tilde{S}_{j6}a_3+S_{j3}A_2 \quad (32b)$$
$$G_{j3} = 2\tilde{S}_{j1}a_7+2\tilde{S}_{j2}a_5-\tilde{S}_{j4}a_{11}+\tilde{S}_{j5}a_{12}-\tilde{S}_{j6}a_6+S_{j3}A_3 \quad j = 1,2,4,5,6. \quad (32c)$$

The coefficients in eqn (28) are determined by satisfaction of the governing equations (10), the traction-free boundary conditions, eqn (12), and the interfacial conditions (14), (15). To this end, a system of 34 linear algebraic equations is obtained for the 44 unknown coefficients in eqns (29)–(32). Equation (10) yields

$$-6\tilde{S}_{24}a_1+2(\tilde{S}_{25}+\tilde{S}_{46})a_2-2(\tilde{S}_{14}+\tilde{S}_{56})a_3+6\tilde{S}_{15}a_4+\tilde{S}_{44}a_8$$
$$-2\tilde{S}_{45}a_9+2\tilde{S}_{55}a_{10} = -2A_4+A_1S_{34}-A_2S_{35} \quad (33a)$$

$$a_i^{(m),(m+1)} = 0 \quad i = 3,4,6,7,10,12 \quad (33b)$$

and the following quantities are identical for the m and the $m+1$ plies:

$$a_j,\quad j=1,2,5,8,11;\quad A_jS_{33},\quad i=1,2,3;\quad A_4;\quad G_{11};\quad G_{13};\quad G_{61}-G_{12};$$
$$\tfrac{1}{2}G_{63}+\omega_3;\quad G_{51};\quad G_{53};\quad u_{i0},\quad i=1,2,3;\quad \omega_k,\quad k=1,2. \quad (33c\text{–}m)$$

Using eqns (33b,c), we can rewrite (33a) in the form

$$(\tilde{S}^{(k)}_{25}+\tilde{S}^{(k)}_{16})a_2+\tilde{S}^{(k)}_{44}a_8+2A_4=0 \quad k=m,m+1. \tag{34}$$

Similar elimination will provide

$$G_{11}=6\tilde{S}_{12}a_1+\tilde{S}_{15}a_9+S_{13}A_1 \tag{35a}$$
$$G_{13}=2\tilde{S}_{12}a_5+S_{13}A_3 \tag{35b}$$
$$G_{61}=-2\tilde{S}_{64}a_8-2\tilde{S}_{66}a_2 \tag{35c}$$
$$G_{12}=2\tilde{S}_{12}a_2+S_{13}A_2 \tag{35d}$$
$$G_{63}=-\tilde{S}_{64}a_{11} \tag{35e}$$
$$G_{51}=6\tilde{S}_{52}a_1+\tilde{S}_{55}a_9 \tag{35f}$$
$$G_{53}=2\tilde{S}_{52}a_5. \tag{35g}$$

The use of eqn (35) in (33d–k) assuming no rigid body translations and rotations, results in

$$a_1[6(\tilde{S}^{(m)}_{12}-\tilde{S}^{(m+1)}_{12})]+a_9[\tilde{S}^{(m)}_{15}]-a'_9[\tilde{S}^{(m+1)}_{15}]+A_1[S^{(m)}_{13}-S^{(m+1)}_{13}S^{(m)}_{33}/S^{(m+1)}_{33}]=0 \tag{36}$$

$$a_5[2(\tilde{S}^{(m)}_{12}-\tilde{S}^{(m+1)}_{12})]+A_3[S^{(m)}_{13}-S^{(m+1)}_{13}S^{(m)}_{33}/S^{(m+1)}_{33}]=0 \tag{37}$$

$$a_8[2(\tilde{S}^{(m)}_{64}-\tilde{S}^{(m+1)}_{64})]+a_2[2(\tilde{S}^{(m)}_{66}+\tilde{S}^{(m)}_{12}-\tilde{S}^{(m+1)}_{66}-\tilde{S}^{(m+1)}_{12})]$$
$$+A_2[S^{(m)}_{13}-S^{(m+1)}_{13}S^{(m)}_{33}/S^{(m+1)}_{33}]=0 \tag{38}$$

$$a_{11}[\tilde{S}^{(m)}_{64}-\tilde{S}^{(m+1)}_{64}]=0 \tag{39}$$

$$a_1[6(\tilde{S}^{(m)}_{52}-\tilde{S}^{(m+1)}_{52})]+a_9[\tilde{S}^{(m)}_{55}]-a'_9[\tilde{S}^{(m+1)}_{55}]=0 \tag{40}$$

$$a_5[2(\tilde{S}^{(m)}_{52}-\tilde{S}^{(m+1)}_{52})]=0. \tag{41}$$

At this stage, we are left with 11 unknowns: a_1, a_2, a_5, a_8, a_9, a'_9, a_{11}, A_1, A_2, A_3, A_4 where a_9 and a'_9 are for the mth and $(m+1)$th plies, respectively, and the rest of the coefficients are identical for both layers. These unknowns appear in the eight equations, (34), (36)–(41). In order to impose the far-end conditions (13), the full expressions for the stresses (i.e. the sum of the homogeneous and particular parts) are needed. This adds the inifinite number of unknowns $d^{(h)}_n$ [eqn (27)]. It should be noted that although in practical computation this set of unknowns, $d^{(h)}_n$, is truncated, the system of equations is still over determined since some of the unknowns were eliminated by the additional equations. Next, the double integrals in (13) are exactly evaluated. The results are in the Appendix. Equations (A1)–(A6) together with (34), (36)–(41), form a system of 14 equations in the above 11 unknowns and the additional unknowns $\{d_n\}$.

In order to incorporate the effect of the hole, the approach mentioned in (2.2.3) is applied. It should be noted that the calculated stresses using Lekhnitskii's theory for anisotropic plate with elliptical cavity are $\sigma^{(m)}_{13\text{cal}}$ and $\sigma^{(m)}_{33\text{cal}}$. Thus,

$$\sigma_{33}^{(m)}(x_1,x_2) = \sigma_{33\,\mathrm{cal}}^{(m)}(x_1,x_2)$$
$$\sigma_{13}^{(m)}(x_1,x_2) = \sigma_{33\,\mathrm{cal}}^{(m)}(x_1,x_2) \quad m = 1,2,\ldots \text{No. of plies.} \tag{42}$$

The traction-free boundary conditions at the upper and lower surfaces of the laminate are:

$$\sigma_{22}^{(k)}(x_1,x_2) = 0$$
$$\sigma_{23}^{(k)}(x_1,x_2) = 0$$
$$\sigma_{12}^{(k)}(x_1,x_2) = 0 \quad k = 1,2; \quad x_2 = b, -c, \text{ respectively.} \tag{43}$$

At the exterior free edge

$$\sigma_{11}^{(k)}(x_1,x_2) = 0$$
$$\sigma_{13}^{(k)}(x_1,x_2) = 0$$
$$\sigma_{12}^{(k)}(x_1,x_2) = 0 \quad k = 1,2; \quad x_2 = a. \tag{44}$$

For the symmetric laminate, the following relations are required at the plane of symmetry

$$u_{1,2}(x_1,x_2) = 0$$
$$u_{2,1}(x_1,x_2) = 0$$
$$u_{3,2}(x_1,x_2) = 0. \tag{45}$$

The above conditions (42)–(45), are satisfied by minimization of the error of the residuals in the sense of a weighting function technique. The stresses and the relevant derivatives of the displacements [as required in eqn (45)] have the general form

$$\mathbb{R} = \sum_n D_n \phi_n - f \tag{46}$$

where D_n are the coefficients a_i, A_j, $d_k^{(h)}$, and ϕ_n are the trial functions to be identified with the eigenfunctions of the exact solution. The function f is either zero or consists of the solutions obtained from the hole effect, eqn (42). Orthogonalization of $\mathbb{R}$ with the trial functions, such that the inner product vanishes, is performed in the form

$$\left(\phi_j, \left[\sum_{i=1}^{n} D_i\phi_i - f\right]\right) = 0 \quad j = 1,2,\ldots,n. \tag{47}$$

The inner products yields n equations where n is the number of all unknowns taken into account. Thus

$$\sum_{i=1}^{n} D_i \int_{\mathbb{D}} (\phi_j\phi_i)\,\mathrm{d}S = \int_{\mathbb{D}} (\phi_j f)\,\mathrm{d}S \quad j = 1,2,\ldots,n \tag{48}$$

where $\mathbb{D}$ is the domain in which the problem is treated and therefore it is where integration is performed. In our case, this domain is changed according to the line where the boundary condition takes place and that is where integration is carried out. The explicit form of eqn (48) is given by

$$D_i\left[\int_0^a (F_{2i}^{(1)}F_{2j}^{(1)}+F_{4i}^{(1)}F_{4j}^{(1)}+F_{6i}^{(1)}F_{6j}^{(1)})\,dS\Big|_{x_2=b}+\int_0^b (F_{1i}^{(1)}F_{1j}^{(1)}+F_{5i}^{(1)}F_{5j}^{(1)}+F_{6i}^{(1)}F_{6j}^{(1)})\,dS\Big|_{x_2=a}\right.$$
$$+\int_{-c}^{0} (F_{1i}^{(2)}F_{1j}^{(2)}+F_{5i}^{(2)}F_{5j}^{(2)}+F_{6i}^{(2)}F_{6j}^{(2)})\,dS\Big|_{x_1=a}+\int_t^{a-t} (F_{3i}^{(1)}F_{3j}^{(1)}+F_{5i}^{(1)}F_{5j}^{(1)})\,dS\Big|_{x_2=0}$$
$$\left.+\int_t^{a-t} (F_{3i}^{(2)}F_{3j}^{(2)}+F_{5i}^{(2)}F_{5j}^{(2)})\,dS\Big|_{x_2=0}+\int_0^a (L_{1i}^{(2)}L_{1j}^{(2)}+L_{2i}^{(2)}L_{2j}^{(2)}+L_{3i}^{(2)}L_{3j}^{(2)})\,dS\Big|_{x_2=CL}\right]$$
$$=\int_t^{a-t} (\sigma_{3\,\mathrm{cal}}^{(1)}F_{3j}^{(1)}+\sigma_{5\,\mathrm{cal}}^{(1)}F_{5j}^{(1)}+\sigma_{3\,\mathrm{cal}}^{(2)}F_{3j}^{(2)}+\sigma_{5\,\mathrm{cal}}^{(2)}F_{5j}^{(2)})\,dS\Big|_{x_2=0} \tag{49}$$

in which $j = 1, 2, \ldots$ (No. of unknowns) and t is the laminate thickness. In eqn (49), $F_{\alpha i}(m)$ is defined by the homogeneous and particular parts of the solution, eqns (27) and (29) respectively.

The index $\alpha = 1, 2, 3, 4, 5, 6$ denotes the contracted Greek notation for the stresses. The functions $L_{\beta i}(m)$ ($\beta = 1, 2, 3$) are defined by the derivatives (45). The integrals in (49) are performed numerically using Simpson's method. The number of points of integration is of great importance when convergence of the solution is considered. Equations (34), (36)–(41), (49) provide a set of linear algebraic equations

$$AD = \mathbb{B} \tag{50}$$

in which A is the coefficients matrix with order $(q\times n)$, $q > n$; q is the number of unknowns associated with the 14 equations from the elasticity solution (34), (36)–(41), (49), (A1)–(A6), and n is the total number of unknowns. In eqn (49) D is a vector $(n\times 1)$ of the unknowns.

The over determined system may be solved in the least square sense as

$$A^{\dagger}AD = A^{\dagger}\mathbb{B} \tag{51}$$

in which $A^{\dagger}$ is the conjugate transpose of A. Equation (51) turns out to be solvable since $A^{\dagger}A$ is of the order $(n\times n)$ and $A\mathbb{B}^{\dagger}$ is $(n\times 1)$, but due to the nature of the general solution it appears that some rows and/or columns might be zero or show dependency which causes $A^{\dagger}A$ to be singular. Even if a mathematical singularity does not occur, due to the use of a computer, the solution of such a system might be strongly ill-conditioned, depending upon the properties of the plies which enter the equations. This ill-conditioned behavior may be treated by adopting the method of singular-value decomposition (Stewart, 1973). According to this method, every matrix A $(q\times n)$, $q > n$, may be expressed as the multiplication of three matrices as follows

$$A = U\Sigma V^T \tag{52}$$

where U and V are $(q\times q)$ and $(n\times n)$ unitary matrices whose columns are the orthonormalized eigenvectors of AA^T and A^TA, respectively. The matrix Σ is diag $(\sigma, 0)$ which is a $(q\times n)$ matrix with σ being the square roots of the non-zero eigenvalues of A^TA. Let U' be denoted by

$$U' = AV\begin{bmatrix}\Sigma_r^{-1} & \vdots & 0\\ \cdots\cdots & \vdots & \cdots\cdots\\ 0 & \vdots & 0\end{bmatrix} \tag{53}$$

in which Σ_r^{-1} are the reciprocals of the non-zero components of Σ in a descending order on the diagonal. Let us also define w by

$$w = U'^T\mathbb{B} \tag{54}$$

and

$$v_i = \frac{w_i}{\sigma_i} \quad i = 1, 2, \ldots, n. \tag{55}$$

The desired solution is determined from

$$\{D\} = [V]\{v\}. \tag{56}$$

Having obtained $\{D\}$, the displacements and stresses are computed from eqns (27), (29) and (30).

4. CONVERGENCE OF THE SOLUTION AND EXAMPLES

Consider the case of a $[\pm 45]_s$ laminate which was considered by Wang and Choi (1982a,b), Wang and Crossman (1977), Pipes and Pagano (1970) and Kassapoglou and Lagace (1987). The properties of the unidirectional single ply as given by the above mentioned authors, for the graphite–epoxy system are given in Table 1. As a case study, a circular hole is centrally located, and the laminate is subjected to a unit stress in the x_3 direction as shown in Fig. 3(a).

Results were obtained in three locations: in the vicinity of the hole and at the free edge of the laminate as well as far from these two locations where classical lamination theory or results of anisotropic plate analysis with a center hole are valid. Convergence of the obtained stresses was studied by examining the effect of the number of eigenvalues and the number of integration points on the results. The study of the effect of the number of eigenvalues was limited to the ability of the computer to provide accurate solutions when using the Muller (1956) deflation method since this method involves calculations of differences between numbers that converge to the point that multiplication by that difference results in computer underflow. Convergence was studied on all stresses. Predicted results for the normal stress σ_{22} are exhibited for three different numbers of eigenvalues and 200 integration points as shown in Fig. 2. Observing curves 1–5 in Fig. 2, we conclude that convergence is achieved using 25 eigenvalues with slight changes between the cases of 50, 100, and 200 integration points. Curves 4 and 5 show results in which low numbers of eigenvalues are used and therefore result in wrong stress distribution. Curves 1–3 present close results in which the same number of 25 eigenvalues are used and show convergence. The results of curves 1–3 at $x_1/a = 1$, match the results presented by Wang and Choi (1982a,b) for the similar case of a straight free edge. In order to investigate the effect of the hole on the stress distribution at various locations along its circumference, we present in Fig. 3(b) all stress distributions along a cut made perpendicular to the laminate straight free edge, and in Fig. 3(c), the variation of the normal stress σ_{22} along the cross sections which are radial to the hole at $\alpha = 0°$, $10°$, $30°$, $60°$, and $80°$. This figure exhibits well the fiber orientation depend-

Table 1. Material properties of single unidirectional ply of the examined laminate

	E(msi)	SI(GN/m^2)
E_1	20.0	137.50
E_2	2.1	14.44
E_3	2.1	14.44
G_{12}	0.85	5.84
G_{13}	0.85	5.84
G_{23}	0.85	5.84
ν_{12}	0.21	
ν_{13}	0.21	
ν_{23}	0.21	

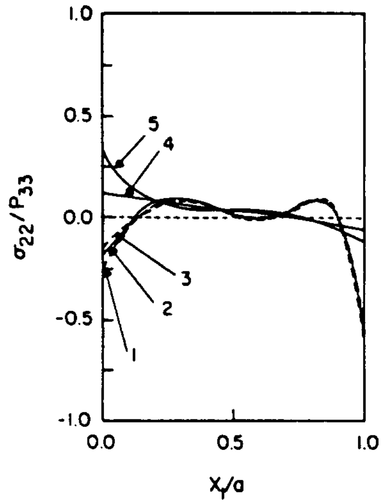

Fig. 2. Convergence of σ_{22} through various numbers of eigenvalues and points of integration.

Curve No.	No. of eigenvalues	Points of integration
1	25	200
2	25	100
3	25	50
4	15	100
5	7	100

ence which provide various orders of singularities. The values of the orders of singularity are given by

$$\delta_1 = -0.025575658 \quad \text{for } \alpha = 0^\circ$$

$$\delta_1 = -0.026100409 \quad \text{for } \alpha = 10^\circ$$

$$\delta_1 = -0.030274706 \quad \text{for } \alpha = 30^\circ$$

$$\delta_1 = -0.030274706 \quad \text{for } \alpha = 60^\circ$$

$$\delta_1 = -0.026100409 \quad \text{for } \alpha = 80^\circ$$

where δ_1 for the cases of $\alpha = 10^\circ, 30^\circ, 60^\circ$ and 80° is calculated for the direction tangential to the hole edge which provides an interface between [35/−55], [15/−75], [−15, 75] and [−35/55] orientations, respectively. The latter analysis is done using a coordinate system transformed to match the direction tangential to the hole and to the radial cross section in order to satisfy the basic assumptions introduced in Section 2.1 and the traction-free boundary conditions at the hole edge as explained in Section 2.2.1. The validity of the results for the transformed configuration are limited to the vicinity of the hole within the region in which the stresses are controlled by the mathematical singularity.

5. CONCLUSIONS

The present investigation exhibits the following points of interest.

(i) σ_{13} satisfies the traction-free boundary condition as also indicated by Kassapogolou and Lagace (1987).

(ii) σ_{22} exhibits singular behavior and tends to $-\infty$ as previously shown by Wang and Choi (1982a,b) for the cross section at $x_{33} = 0$. Note that at this cross section the distribution at the hole is somewhat different from the one at the laminate free edge. That indicates the influence of the hole constraints on the edge effect. However, when examining the hole circumference, we find a region in which σ_{22} is in tension and more likely to cause

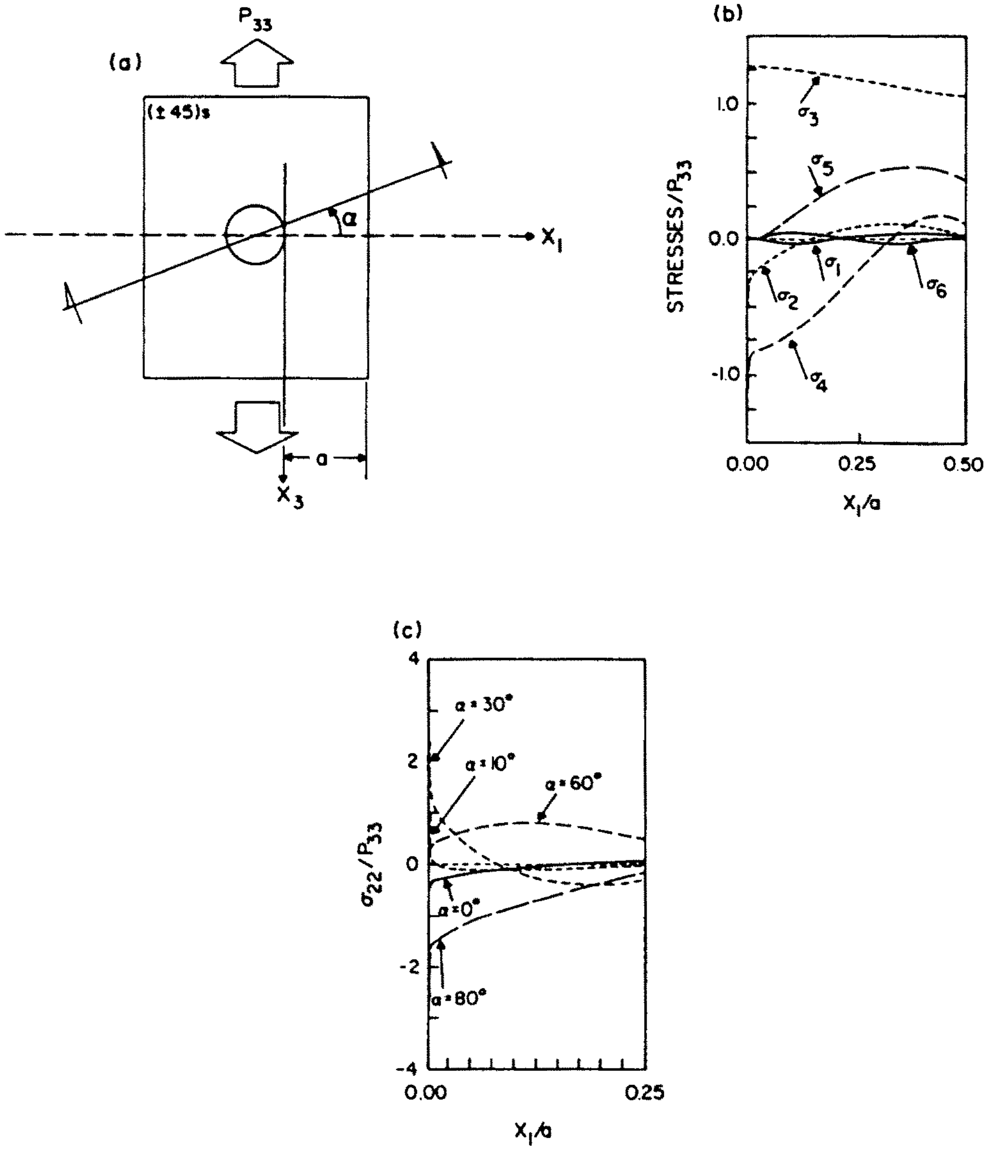

Fig. 3. (a) Geometry and description of cuts in (b) and (c). (b) Stresses distribution at $\alpha = 0°$; 25 eigenvalues, 200 points of integration. (c) Comparison of normal stress σ_{22} at various cuts.

delamination. For the $(\pm 45)_s$ laminate, that region was found to be the arc which lies between 10 and 50 degrees approximately.

(iii) The stress concentration around the hole matches Lekhnitskii's development even though a slight relief is detected at the edge due to the boundary layer effect.

(iv) The present work provides a general framework for the analysis of the r^δ singularity for any type of laminate containing an elliptical cavity, and for the determination of singular stress fields associated with edge effects in those configurations.

This method of analysis has been used successfully to predict certain aspects of delamination in notched composite laminates. These results will be presented in a subsequent paper.

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APPENDIX: INTEGRATION OF THE FAR END CONDITIONS EQUATIONS

$$a_9\left[\frac{x_1^2x_2}{2}\right]+\sum_n d_n\left\{\sum_{k=1}^{3}\left[C_k\eta_k\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}+C_{k+3}\bar{\eta}_k\frac{\bar{Z}_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}\right]\right\}=0 \tag{A1}$$

$$a_8[x_1^2x_2]+a_9\left[\frac{x_1x_2^2}{2}\right]+a_{11}[x_1x_2]+\sum_n d_n\left\{\sum_{k=1}^{3}\left[C_k\eta_k\frac{Z_k^{\delta_n+2}}{\mu_k(\delta_n+1)(\delta_n+2)}+C_{k+3}\bar{\eta}_k\frac{\bar{Z}_k^{\delta_n+2}}{\bar{\mu}_k(\delta_n+1)(\delta_n+2)}\right]\right\}=0 \tag{A2}$$

$$\begin{aligned}
&A_1\left[\frac{x_1^2x_2}{2}\right]+A_2\left[\frac{x_1x_2^2}{2}\right]+A_3[x_1x_2]+a_1\left[-\frac{S_{32}}{S_{33}}3x_1^2x_2\right]+a_2\left[-\frac{S_{32}}{S_{33}}x_2^2x_1+\frac{S_{36}}{S_{33}}x_1^2x_2\right]\\
&+a_5\left[-\frac{S_{32}}{S_{33}}2x_1x_2\right]+a_8\left[\frac{S_{34}}{S_{33}}x_1^2x_2\right]+a_9\left[\frac{S_{34}}{S_{33}}\frac{x_2^2x_1}{2}-\frac{S_{35}}{S_{33}}\frac{x_1^2x_2}{2}\right]+a_{11}\left[\frac{S_{34}}{S_{33}}x_1x_2\right]\\
&+\sum_n d_n\left\{-\frac{S_{31}}{S_{33}}\sum_{k=1}^{3}\left[C_k\mu_k\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}+C_{k+3}\bar{\mu}_k\frac{\bar{Z}_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}\right]\right.\\
&-\frac{S_{32}}{S_{33}}\sum_{k=1}^{3}\left[C_k\frac{Z_k^{\delta_n+2}}{\mu_k(\delta_n+1)(\delta_n+2)}+C_{k+3}\frac{\bar{Z}_k^{\delta_n+2}}{\bar{\mu}_k(\delta_n+1)(\delta_n+2)}\right]\\
&+\frac{S_{34}}{S_{33}}\sum_{k=1}^{3}\left[C_k\frac{\eta_k}{\mu_k}\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}+C_{k+3}\frac{\bar{\eta}_k}{\bar{\mu}_k}\frac{\bar{Z}_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}\right]\\
&-\frac{S_{35}}{S_{33}}\sum_{k=1}^{3}\left[C_k\eta_k\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}+C_{k+3}\bar{\eta}_k\frac{\bar{Z}_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}\right]\\
&\left.+\frac{S_{36}}{S_{33}}\sum_{k=1}^{3}\left[C_k\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}+C_{k+3}\frac{\bar{Z}_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}\right]\right\}=P_{11}
\end{aligned} \tag{A3}$$

$$\begin{aligned}
&A_1\left[\frac{x_1^2x_2^2}{4}\right]+A_2\left[\frac{x_1x_2^3}{3}\right]+A_3\left[\frac{x_1x_2^2}{2}\right]+a_1\left[-\frac{S_{32}}{S_{33}}\frac{3x_1^2x_2^2}{2}\right]+a_2\left[-\frac{S_{32}}{S_{33}}\frac{2x_1x_2^3}{3}+\frac{S_{36}}{S_{33}}\frac{x_1x_2^2}{2}\right]\\
&+a_5\left[-\frac{S_{32}}{S_{33}}x_1x_2^2\right]+a_8\left[\frac{S_{34}}{S_{33}}\frac{x_1^2x_2^2}{2}\right]+a_9\left[\frac{S_{34}}{S_{33}}\frac{x_2^3x_1}{3}-\frac{S_{35}}{S_{33}}\frac{x_1^2x_2^2}{4}\right]+a_{11}\left[\frac{S_{34}}{S_{33}}\frac{x_1x_2^2}{2}\right]\\
&+\sum_n d_n\left\{\sum_{k=1}^{3}\left[C_k\left(\frac{x_2}{\mu_k}\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}-\frac{1}{\mu_k^2}\frac{Z_k^{\delta_n+3}}{(\delta_n+1)(\delta_n+2)(\delta_n+3)}\right)\right.\right.\\
&\times\left(-\frac{S_{31}}{S_{33}}\mu_k^2-\frac{S_{32}}{S_{33}}+\frac{S_{34}}{S_{33}}\eta_k-\frac{S_{35}}{S_{33}}\eta_k\mu_k+\frac{S_{36}}{S_{33}}\mu_k\right)\\
&C_{k+3}\left(\frac{x_2}{\bar{\mu}_k}\frac{\bar{Z}_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}-\frac{1}{\bar{\mu}_k^2}\frac{\bar{Z}_k^{\delta_n+3}}{(\delta_n+1)(\delta_n+2)(\delta_n+3)}\right)\\
&\left.\left.\times\left(-\frac{S_{31}}{S_{33}}\bar{\mu}_k^2-\frac{S_{32}}{S_{33}}+\frac{S_{34}}{S_{33}}\bar{\eta}_k-\frac{S_{35}}{S_{33}}\bar{\eta}_k\bar{\mu}_k+\frac{S_{36}}{S_{33}}\bar{\mu}_k\right)\right]\right\}=M_{11}
\end{aligned} \tag{A4}$$

$$\begin{aligned}
&A_1\left[\frac{x_1^3x_2}{3}\right]+A_2\left[\frac{x_1^2x_2^2}{4}\right]+A_3\left[\frac{x_1^2x_2}{2}\right]+a_1\left[-\frac{S_{32}}{S_{33}}2x_1^3x_2\right]+a_2\left[-\frac{S_{32}}{S_{33}}\frac{x_1x_2^2}{2}+\frac{S_{36}}{S_{33}}\frac{2x_1x_2^2}{3}\right]\\
&+a_5\left[-\frac{S_{32}}{S_{33}}x_1^2x_2\right]+a_8\left[\frac{S_{34}}{S_{33}}\frac{2x_1^3x_2}{3}\right]+a_9\left[\frac{S_{34}}{S_{33}}\frac{x_1^2x_2^2}{4}-\frac{S_{35}}{S_{33}}\frac{x_1^3x_2}{3}\right]+a_{11}\left[\frac{S_{34}}{S_{33}}\frac{x_1^2x_2}{2}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_n d_n\left\{\sum_{k=1}^{3}\left[C_k\left(\frac{x_1}{\mu_k}\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}-\frac{1}{\mu_k}\frac{Z_k^{\delta_n+3}}{(\delta_n+1)(\delta_n+2)(\delta_n+3)}\right)\right.\right.\\
&\quad\times\left(-\frac{S_{31}}{S_{33}}\mu_k^2-\frac{S_{32}}{S_{33}}+\frac{S_{34}}{S_{33}}\eta_k-\frac{S_{35}}{S_{33}}\eta_k\mu_k-\frac{S_{36}}{S_{33}}\mu_k\right)\\
&\quad+C_{k+3}\left(\frac{x_1}{\bar{\mu}_k}\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}-\frac{1}{\bar{\mu}_k}\frac{Z_k^{\delta_n+3}}{(\delta_n+1)(\delta_n+2)(\delta_n+3)}\right)\\
&\quad\left.\left.\times\left(-\frac{S_{31}}{S_{33}}\bar{\mu}_k^2-\frac{S_{32}}{S_{33}}+\frac{S_{34}}{S_{33}}\bar{\eta}_k-\frac{S_{35}}{S_{33}}\bar{\eta}_k\bar{\mu}_k+\frac{S_{36}}{S_{33}}\bar{\mu}_k\right)\right]\right\}=M_{22}
\end{aligned}
\qquad \text{(A5)}
$$

$$
\begin{aligned}
&a_8\left[\frac{2x_1^3x_2}{3}\right]+a_9\left[\frac{x_1^2x_2^2}{2}\right]+a_{11}\left[\frac{x_1^2x_2}{2}\right]+\sum_n d_n\left\{\sum_{k=1}^{3}\left[C_k\eta_k\left(\left[\frac{x_1}{\mu_k}+x_2\right]\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}-\frac{1}{\bar{\mu}_k}\frac{Z_k^{\delta_n+3}}{(\delta_n+1)(\delta_n+2)(\delta_n+3)}\right)\right.\right.\\
&\qquad\left.\left.+C_{k+3}\bar{\eta}_k\left(\left[\frac{x_1}{\bar{\mu}_k}+x_2\right]\frac{Z_k^{\delta_n+2}}{(\delta_n+1)(\delta_n+2)}-\frac{1}{\bar{\mu}_k}\frac{Z_k^{\delta_n+3}}{(\delta_n+1)(\delta_n+2)(\delta_n+3)}\right)\right]\right\}=-M_{12}
\end{aligned}
\qquad \text{(A6)}
$$